

PROPERTIES OF PFAFFIANS,
WITH THEIR ANALOGUES IN DETERMINANTS.

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1. In the course of a recent inquiry connected with Pfaffians the identity

$$\begin{aligned} & -15|2346| + 16|2345| + 56|1234| \\ & \quad = 23|1456| - 24|1356| + 34|1256| \end{aligned}$$

or, in more concrete notation,

$$\begin{aligned} & -a_3 \begin{vmatrix} b_3 & b_6 \\ c_4 & c_6 \\ d_6 & \end{vmatrix} + a_6 \begin{vmatrix} b_3 & b_5 \\ c_4 & c_5 \\ d_5 & \end{vmatrix} + e_6 \begin{vmatrix} a_2 & a_3 & a_4 \\ & b_3 & b_4 \\ & c_3 & c_4 \end{vmatrix} \\ & \quad = b_3 \begin{vmatrix} a_4 & a_5 & a_6 \\ & d_5 & d_6 \\ & e_5 & e_6 \end{vmatrix} - b_4 \begin{vmatrix} a_3 & a_5 & a_6 \\ & c_5 & c_6 \\ & e_5 & e_6 \end{vmatrix} + c_4 \begin{vmatrix} a_2 & a_5 & a_6 \\ & b_5 & b_6 \\ & e_5 & e_6 \end{vmatrix} \end{aligned}$$

made its appearance, and had to be verified. It is most readily established by noting that the Pfaffians occurring in it are principal minors of

$$|123456| \quad \text{or} \quad |a_2 b_3 c_4 d_5 e_6|,$$

and that therefore the two members of the identity are by a theorem of 1903* each equal to

$$|a_2 b_3 c_4 d_5 e_6| - |a_4 b_5 c_6|.$$

There is, however, another way of looking at it which is more instructive and which leads with ease to other identities of like kind.

2. Denoting the said Pfaffian of the third degree by f , and its adjugate by

$$|A_2 B_3 C_4 D_5 E_6|,$$

* *Trans. of S. African Philos. Soc.*, xv., pp. 35-41.

we have, of course

$$\left. \begin{aligned} \ddot{f} &= a_2 A_2 + a_3 A_3 + a_4 A_4 + a_5 A_5 + a_6 A_6 \\ \ddot{f} &= a_2 A_2 + b_3 B_3 + b_4 B_4 + b_5 B_5 + b_6 B_6 \\ \ddot{f} &= a_3 A_3 + b_3 B_3 + c_4 C_4 + c_5 C_5 + c_6 C_6 \\ \ddot{f} &= a_4 A_4 + b_4 B_4 + c_4 C_4 + d_5 D_5 + d_6 D_6 \\ \ddot{f} &= a_5 A_5 + b_5 B_5 + c_5 C_5 + d_5 D_5 + e_6 E_6 \\ \ddot{f} &= a_6 A_6 + b_6 B_6 + c_6 C_6 + d_6 D_6 + e_6 E_6 \end{aligned} \right\},$$

where the oblong assemblage of terms on the right has a peculiar approximation to axisymmetry and can be made actually axisymmetric by introducing a zero diagonal. Subtracting now any one of the six expressions, say the first, from the sum of the remaining five we obtain

$$\begin{aligned} 4\ddot{f} &= 2(b_3 B + b_4 B_4 + \dots + e_6 E_6), \\ \text{and } \therefore 2\ddot{f} &= (b_3 \ b_4 \ b_5 \ b_6 \ \check{B}_3 \ B_4 \ B_5 \ B_6), \\ &\quad \begin{array}{c|c} c_4 \ c_5 \ c_6 & C_4 \ C_5 \ C_6 \\ d_5 \ d_6 & D_5 \ D_6 \\ e_6 & E_6 \end{array} \end{aligned}$$

and five others similar to it. Again, by subtracting the sum of any two, say the first and second, from the sum of the remaining four, and dividing by 2 we find

$$\ddot{f} = (c_4 \ c_5 \ c_6 \ \check{C}_4 \ C_5 \ C_6) - a_2 A_2, \\ \begin{array}{c|c} d_5 \ d_6 & D_5 \ D_6 \\ e_6 & E_6 \end{array}$$

and fourteen others similar to it. And, lastly, by subtracting the sum of any three from the sum of the remaining three we obtain

$$0 = (d_5 \ d_6 \ \check{D}_5 \ D_6) - (a_2 \ a_3 \ \check{A}_2 \ A_3) \\ \begin{array}{c|c} e_6 & E_6 \\ b_3 & B_3 \end{array}$$

and nineteen others similar to it. Among the nineteen is the identity with which we started.

The general theorem may be enunciated thus: If μ , ν be complementary minors of a Pfaffian, and M, N the corresponding minors of the adjugate, then

$$(\mu \check{M}) - (\nu \check{N})$$

is a multiple of the Pfaffian, the multiplier being 0 when μ , ν are of the same order.

3. In determinants there is an almost perfectly analogous theorem. Here, however, instead of having one set of expressions, all of which are

employed in reaching any case of the theorem, we now have two sets, the one arising from consideration of the rows and the other from the columns. For example, the determinant being

$$|a_1 b_2 c_3 d_4 e_5 f_6|, \text{ or } \Delta \text{ say,}$$

we have for it the six expressions

$$\begin{aligned} & a_1 A_1 + a_2 A_2 + \dots + a_6 A_6, \\ & b_1 B_1 + b_2 B_2 + \dots + b_6 B_6, \\ & \dots\dots\dots \end{aligned}$$

and the other six

$$\begin{aligned} & a_1 A_1 + b_1 B_1 + \dots + f_1 F_1 \\ & a_2 A_2 + b_2 B_2 + \dots + f_2 F_2 \\ & \dots\dots\dots, \end{aligned}$$

and our procedure now is to subtract the sum of any number of the one set from the sum of any number of the other set. Thus, if in the first set the expressions summed be the 2nd and 3rd, and in the second set they be the 1st, 5th, 6th, the result of subtraction is

$$\Delta = (a_1 \ a_5 \ a_6 | A_1 \ A_5 \ A_6) - (b_2 \ b_3 \ b_4 | B_2 \ B_3 \ B_4) \\ \left| \begin{array}{ccc|ccc} d_1 & d_5 & d_6 & D_1 & D_5 & D_6 \\ e_1 & e_5 & e_6 & E_1 & E_5 & E_6 \\ f_1 & f_5 & f_6 & F_1 & F_5 & F_6 \end{array} \right| \left| \begin{array}{ccc|ccc} c_2 & c_3 & c_4 & C_2 & C_3 & C_4 \end{array} \right|$$

When the number of expressions taken from the one set together with the number taken from the other amount to 6, the arrays in the result besides being complementary are square: and when the number taken from both is 3, the coefficient of Δ is 0.

4. The set of six expressions for $\mathcal{P}$ in § 2 is of course the analogue of the two sets of expressions for Δ in § 3. The latter two sets, however, have associated with them two other well-known sets, namely,

$$\left. \begin{aligned} & a_1 B_1 + a_2 B_2 + \dots + a_6 B_6 = 0 \\ & a_1 C_1 + a_2 C_2 + \dots + a_6 C_6 = 0 \\ & \dots\dots\dots \end{aligned} \right\}$$

and

$$\left. \begin{aligned} & a_1 A_2 + b_1 B_2 + \dots + f_1 F_2 = 0 \\ & a_1 A_3 + b_1 B_3 + \dots + f_1 F_3 = 0 \\ & \dots\dots\dots \end{aligned} \right\},$$

and when we seek for the analogues of these in the theory of Pfaffians, they are not so readily arrived at. It is interesting therefore to note that

they make their appearance when we try to establish the relation between a Pfaffian and its adjugate.

From the theory of determinants we know that the square of this adjugate is

$$\begin{vmatrix} . & A_2 & A_3 & A_4 & A_5 & A_6 \\ -A_2 & . & B_3 & B_4 & B_5 & B_6 \\ -A_3 & -B_3 & . & C_4 & C_5 & C_6 \\ -A_4 & -B_4 & -C_4 & . & D_5 & D_6 \\ -A_5 & -B_5 & -C_5 & D_5 & . & E_6 \\ -A_6 & -B_6 & -C_6 & -D_6 & -E_6 & . \end{vmatrix}$$

where

$$A_2 \equiv \begin{vmatrix} c_4 & c_5 & c_6 \\ d_5 & d_6 & e_6 \end{vmatrix}, \quad A_3 \equiv - \begin{vmatrix} b_4 & b_5 & b_6 \\ d_5 & d_6 & e_6 \end{vmatrix},$$

and where, generally, the cofactor of any element of the original Pfaffian $|a_2 b_3 c_4 d_5 e_6|$ is got by deleting the two frame-lines in which the element is situated and prefixing the sign - when the sum of the numbers of the lines is even. Multiplying this determinant row-wise by the determinant which is the square of $|a_2 b_3 c_4 d_5 e_6|$ itself, namely, by

$$\begin{vmatrix} . & a_2 & a_3 & a_4 & a_5 & a_6 \\ -a_2 & . & b_3 & b_4 & b_5 & b_6 \\ -a_3 & -b_3 & . & c_4 & c_5 & c_6 \\ -a_4 & -b_4 & -c_4 & . & d_5 & d_6 \\ -a_5 & -b_5 & -c_5 & d_5 & . & e_6 \\ -a_6 & -b_6 & -c_6 & -d_6 & -e_6 & . \end{vmatrix}$$

we obtain

$$\begin{vmatrix} \mathcal{f}\mathcal{f} & . & . & . & . & . \\ . & \mathcal{f}\mathcal{f} & . & . & . & . \\ . & . & \mathcal{f}\mathcal{f} & . & . & . \\ . & . & . & \mathcal{f}\mathcal{f} & . & . \\ . & . & . & . & \mathcal{f}\mathcal{f} & . \\ . & . & . & . & . & \mathcal{f}\mathcal{f} \end{vmatrix};$$

in other words, we have the equality

$$(\text{adjug. of } \mathcal{f})^2 \cdot \mathcal{f}^2 = \mathcal{f}^6,$$

from which it follows that

$$\text{adjug. of } \mathcal{f} = \mathcal{f}^{3-1},$$

and, generally, that

$$|A_{1,2n}| = |a_{1,2n}|^{n-1}.$$

Now in the performance here of the row-by-row multiplication we make use of thirty-six results, six of which constitute the set whose analogue we know; and the remaining thirty are those we are in search of, namely,

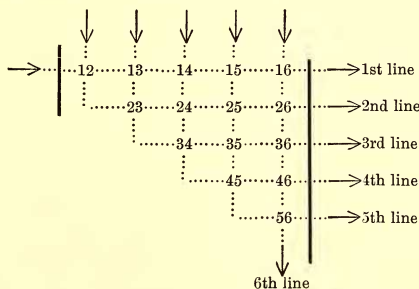
$$\left. \begin{array}{l} (0, A_2, A_3, A_4, A_5, A_6) \not\propto (-a_2, 0, b_3, b_4, b_5, b_6) = 0 \\ \not\propto (-a_3, -b_3, 0, c_4, c_5, c_6) = 0 \\ \dots\dots\dots \\ (-A_6, -B_6, -C_6, -D_6, -E_6, 0) \not\propto (-a_5, -b_5, -c_5, -d_5, 0, e_6) = 0 \end{array} \right\}$$

or, in less instructive form,

$$\left. \begin{aligned} b_3 A_3 + b_4 A_4 + b_5 A_5 + b_6 A_6 &= 0 \\ -b_3 A_2 + c_4 A_4 + c_5 A_5 + c_6 A_6 &= 0 \\ \dots\dots\dots \\ a_5 A_6 + b_5 B_6 + c_5 C_6 + d_5 D_6 &= 0 \end{aligned} \right\}$$

5. On recalling the fact that the vanishing expressions corresponding to these in the theory of determinants are viewable as determinants with two rows or two columns identical we are led to seek for the analogous theorem regarding the nullification of a Pfaffian. It may be stated as follows: *The value of a Pfaffian is 0 if the elements in the part of the rth line which is parallel to a part of the sth line be identical with the corresponding elements of the latter, the element common to the two lines be 0, and the remaining elements of the rth line differ only in sign from the remaining elements of the sth line.* For example, the lines being the 3rd and 5th,* we have

* The following diagram of the frame-lines of the Pfaffian $|12.23.34.45.56|$ will help to make this clear:—



The 3rd and 5th lines are seen to be parallel at the outset and at the close; their common element is 35; and their remaining elements are 34 and 45. (See my Text-book of Determinants, pp. 197-204; *Proc. Lond. Math. Soc.*, xiii., pp. 161-164; and *Transac. R. Soc. Edinburgh*, xl., pp. 49-58).

$$\begin{vmatrix} a_2 & a_5 & a_4 & a_5 & a_6 \\ b_5 & b_4 & b_5 & b_6 \\ & -d_5 & . & e_6 \\ & & d_5 & d_6 \\ & & & e_6 \end{vmatrix} = 0.$$

With this in front of us it is easy to see that the identities at the close of the preceding paragraph can be written in the form

$$0 = \begin{vmatrix} . & b_3 & b_4 & b_5 & b_6 \\ b_3 & b_4 & b_5 & b_6 \\ & c_4 & c_5 & c_6 \\ & & d_5 & d_6 \\ & & & e_6 \end{vmatrix} = \begin{vmatrix} -b_3 & . & c_4 & c_5 & c_6 \\ b_3 & b_4 & b_5 & b_6 \\ & c_4 & c_5 & c_6 \\ & & d_5 & d_6 \\ & & & e_6 \end{vmatrix} = \dots$$

6. As the vanishing of a determinant which has two rows or two columns identical is usually deduced from the fact that a determinant is only altered in sign when two of its rows or columns are interchanged, it is not unnatural that the theorem formulated in the preceding paragraph regarding Pfaffians should be deducible in a similar way. As a matter of fact such is the case, the antecedent theorem being as follows: *The value of a Pfaffian is only altered in sign if the elements in the part of the rth line which is parallel to a part of the sth line be interchanged with the corresponding elements of the latter, the element common to the two lines be changed in sign, and the remaining elements of the rth line be altered in sign and interchanged with the remaining elements of the sth line similarly altered.* For example, the 3rd and 6th lines being taken, it is seen that the parallel parts of them are

$$\begin{matrix} a_3 & \text{and} & a_6 \\ b_3 & & b_6, \end{matrix}$$

the common element is c_6 , and the remaining parts are

$$\begin{matrix} c_4 & c_5 & \text{and} & d_6; \\ & & & e_6, \end{matrix}$$

we thus have

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ b_3 & b_4 & b_5 & b_6 \\ & c_4 & c_5 & c_6 \\ & & d_5 & d_6 \\ & & & e_6 \end{vmatrix} = - \begin{vmatrix} a_2 & a_6 & a_4 & a_5 & a_3 \\ b_6 & b_4 & b_5 & b_3 \\ & -d_6 & -e_6 & -c_6 \\ & & d_5 & -c_4 \\ & & & -c_5 \end{vmatrix}.$$

As the like operation performed with the 1st and 3rd lines of

$$\begin{array}{ccccccc} | -b_3 & . & c_4 & c_5 & c_6 & & \\ & b_3 & b_4 & b_5 & b_6 & & \\ & & c_4 & c_5 & c_6 & & \\ & & & d_5 & d_6 & & \\ & & & & e_6 & & \end{array}$$

leaves it unchanged in form, its value must be 0, as we have seen.

7. Going still a stage further back we have the theorem : *The value of a Pfaffian remains unaltered if the elements in the part of the rth line which is parallel to a part of the sth line be increased by m times the corresponding elements of the latter, the element common to the two lines be left unchanged, and the remaining elements of the rth line be diminished by m times the remaining elements of the sth line in order.* For example, when the two lines are the 2nd and 5th we have

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5 & a_6 \\ b_3 & b_4 & b_5 & b_6 \\ c_4 & c_5 & c_6 \\ d_5 & d_6 \\ e_6 \end{vmatrix} = \begin{vmatrix} a_2+ma_5 & a_3 & a_4 & a_5 & a_6 \\ b_3-mc_5 & b_4-md_5 & b_5 & b_6+me_6 \\ c_4 & c_5 & c_6 \\ d_5 & d_6 \\ e_6 \end{vmatrix},$$

and when the lines are the 5th and 2nd we have it equal to

$$\begin{vmatrix} a_2 & a_3 & a_4 & a_5+ma_2 & a_6 \\ b_3 & b_4 & b_5 & b_6 \\ c_4 & c_5-mb_3 & c_6 \\ d_5-mb_4 & d_6 \\ e_6+mb_6 \end{vmatrix}.$$

The theorem, like its analogue, may be effectively used in making 'evaluations.' Thus

$$\begin{vmatrix} 2 & 3 & 4 & 5 & 6 \\ & 3 & 6 & 10 & 15 \\ & & 4 & 10 & 20 \\ & & & 5 & 15 \\ & & & & 6 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 4 & 5 & 1 \\ & 3 & 6 & 10 & 5 \\ & & 4 & 10 & 10 \\ & & & 5 & 10 \\ & & & & 6 \end{vmatrix} = \dots$$

$$= \begin{vmatrix} . & . & . & . & 1 \\ & 8 & 6 & -3 & 5 \\ & & -6 & -22 & 10 \\ & & & -21 & 10 \\ & & & & 6 \end{vmatrix}$$

$$= \begin{vmatrix} 8 & 6 & -3 \\ -6 & -22 \\ -21 \end{vmatrix} = -168 + 132 + 18 \\ = -18.$$

8. With its help also there may be deduced a condensation-theorem or Pfaffians similar to that for determinants,—that is to say, a theorem expressing a Pfaffian of the n th order by means of one of the $(n-1)$ th order.

Thus, taking $|a_2 b_3 c_4 d_5 e_6 f_7 g_8|$ and performing in succession the operations

$$a_2 \text{line}_3 - a_3 \text{line}_2, \quad a_2 \text{line}_4 - a_4 \text{line}_2, \quad a_2 \text{line}_5 - a_5 \text{line}_2, \dots$$

we change into 0 all the elements of the 1st line except a_2 , and thence derive the result

$$|a_2 b_3 c_4 d_5 e_6 f_7 g_8| = \frac{1}{a_2^2} \begin{vmatrix} |a_2 b_3 c_4| & |a_2 b_3 c_5| & |a_2 b_3 c_6| & |a_2 b_3 c_7| & |a_2 b_3 c_8| \\ |a_2 b_4 d_5| & |a_2 b_4 d_6| & |a_2 b_4 d_7| & |a_2 b_4 d_8| \\ |a_2 b_5 e_6| & |a_2 b_5 e_7| & |a_2 b_5 e_8| \\ |a_2 b_6 f_7| & |a_2 b_6 f_8| \\ |a_2 b_7 g_8| \end{vmatrix}$$

The case of the preceding order,

$$|a_2 b_3 c_4 d_5 e_6| = \frac{1}{a_2} \begin{vmatrix} |a_2 b_3 c_4| & |a_2 b_3 c_5| & |a_2 b_3 c_6| \\ |a_2 b_4 d_5| & |a_2 b_4 d_6| \\ |a_2 b_5 e_6| \end{vmatrix},$$

is not so interesting, because the Pfaffian on the right is then a minor of the adjugate of the Pfaffian on the left, and the identity is thus otherwise known.

If we write the theorem in the umbral notation, for example,

$$|12345678|. (12)^2 = \begin{vmatrix} |1234| & |1235| & |1236| & \dots \\ |1245| & |1246| & \dots \\ \dots \end{vmatrix}$$

we see that it may be viewed as an 'extensional' of the manifest identity

$$|345678| = | \begin{array}{cccccc} 34 & 35 & 36 & 37 & 38 \\ & 45 & 46 & 47 & 48 \\ & & 56 & 57 & 58 \\ & & & 67 & 68 \\ & & & & 78 \end{array} |,$$

—a fact which makes the analogy with the corresponding theorem in determinants still more striking.

9. On the other hand it has to be noted that although in the case of the other known condensation-theorem, Hermite's* of 1849, there is also an analogue, the resemblance is not nearly so close, the elements of the new Pfaffian being no longer Pfaffians themselves but determinants. Thus

$$|a_2 b_3 c_4 d_5 e_6| = \frac{1}{a_3 a_4 a_5} \left| \begin{array}{ccc} . & a_3 & a_4 \\ a_2 & b_3 & b_4 \\ a_3 & . & c_4 \end{array} \right| \left| \begin{array}{ccc} . & a_4 & a_5 \\ a_2 & b_4 & b_5 \\ a_3 & c_4 & c_5 \end{array} \right| \left| \begin{array}{ccc} . & a_5 & a_6 \\ a_2 & b_5 & b_6 \\ a_3 & c_5 & c_6 \end{array} \right|$$

$$\left| \begin{array}{ccc} . & a_4 & a_5 \\ a_3 & c_4 & c_5 \\ a_4 & . & d_5 \end{array} \right| \left| \begin{array}{ccc} . & a_5 & a_6 \\ a_3 & c_5 & d_5 \\ a_4 & c_6 & d_6 \end{array} \right|$$

$$\left| \begin{array}{ccc} . & a_5 & a_6 \\ a_4 & d_5 & d_6 \\ a_5 & . & e_6 \end{array} \right|,$$

and if the elements of the new Pfaffian here be denoted by

$$P_2, P_3, P_4; Q_3, Q_4; R_4$$

the next case may be written

$$|a_2 b_3 c_4 d_5 e_6 f_7 g_8| = \frac{1}{a_2 a_4 a_5 a_6 a_7} |P_2 Q_3 R_4 S_5 T_6|,$$

and similarly for higher orders.

10. There is apparently no theorem in Pfaffians corresponding to the strictly so-called "multiplication-theorem" of determinants. The theorem

* HERMITE, C., "Sur une question relative à la théorie des nombres." *Journ. (de Liouville) de Math.*, xiv., pp. 21-30.

in determinants, however, which concerns the multiplication of an n -line determinant by an expression of n terms has a close analogue in Pfaffians, namely, *The product of an n -line Pfaffian by an expression of n terms is equal to the sum of n Pfaffians the r th of which is got from the given Pfaffian by multiplying the elements of the r th line by the terms of the given expression deprived of its r th term.* For example

$$\begin{vmatrix} a_2 & a_3 & a_4 \\ b_3 & b_4 & c_4 \end{vmatrix} \times (x_1 + x_2 + x_3 + x_4) \\ = \begin{vmatrix} x_2 a_2 & x_3 a_3 & x_4 a_4 \\ b_3 & b_4 & c_4 \end{vmatrix} + \begin{vmatrix} x_1 a_2 & a_3 & a_4 \\ x_3 b_3 & x_4 b_4 & c_4 \end{vmatrix} + \begin{vmatrix} a_2 & x_1 a_3 & a_4 \\ x_2 b_3 & b_4 & x_4 c_4 \end{vmatrix} \\ + \begin{vmatrix} a_2 & a_3 & x_1 a_4 \\ b_3 & x_2 b_4 & x_3 c_4 \end{vmatrix}.$$

By way of proof we have only got to look for the cofactor of x on the right-hand side.